Given $\sum \alpha_i = +\infty$, $\alpha_i > 0$. Show $S_n = \sum X_i \alpha_i$ X_i iid $\in \{$ Uniform.

$\sup S_n = +\infty$, $\inf S_n = -\infty$, a.s.

12.D Le Gall :

$P(\cancel{\max_{i=1..n} S_n}$

$$\int_a^\infty P(X > t)\,dt = \int_a^\infty \int_a^\infty 1_{\{x \geq t\}}\, dP(x)\, dt = \int_a^\infty (x-a)\, dP(x)$$

$$= E[(X-a) 1_{\{X > a\}}]$$

$$E[X 1_{\{X>a\}}] = \int_a^\infty P(X>t)\,dt + aP(X > a)$$

$$t P\left(\max_{i=1..n} S_i^2 - \sum \alpha_i^2 > t\right) \leq E[(S_n^2 - \sum \alpha_i^2)\, 1_{\{M_n > 0\}}]$$

$M_n = S_n^2 - \sum \alpha_i^2$

$$= \int_0^\infty P(S_n^2 - \sum \alpha_i^2 \geq t)\,dt \quad \text{—} \star 1$$

From above

$-\alpha_i \leq \alpha_i X_i \leq \alpha_i$

$b - a = 2\alpha_i$

From Hoeffding $P(S_n \geq t) \leq \exp\left(\frac{-2t^2}{\sum 4\alpha_i^2}\right)$

$\sum_{i=1}^n \alpha_i^2 = a_n$

$$P(S_n^2 \geq t + a_n) \leq \exp\left(-\frac{1}{2}\frac{(t+a_n)}{a_n}\right)$$

$$\star 1 = \int_0^\infty \exp\left(-\frac{1}{2}\frac{(t+a)}{a}\right) dt \qquad \frac{t+a}{2a} = u$$

$$= a\int_{1/2}^\infty \exp(-u)\,du = 0.606\,a$$

$t = 0.8a$

$$\Rightarrow P\left(\sup S_i^2 - \sum_{j=1}^i \alpha_j^2 \geq t\right) \leq \frac{0.606a}{t} \leq \theta$$

$$P\left(\sup S_i^2 - \sum_{j=1}^i \alpha_i^2 \geq t\right) \geq P\left(\sup_i S_i^2 \geq t + \sum_{i=1}^n \alpha_i^2\right)$$

①.

This does not give us anything. So lets try

$$t P\left(\sup_j \sum_{i=1}^{j} \alpha_i^2 - S_j^2 \geq t\right)$$

$$0 = E[M_n] = E[M_n 1_{\{M_n > 0\}}] + E[M_n 1_{\{M_n < 0\}}]$$

$$\leq E\left[(-M_n) 1_{\{M_n < 0\}}\right]$$

$$= E\left[M_n 1_{\{M_n > 0\}}\right] \leq 0.4 \sum_{i=1}^{n} \alpha_i^2$$

Set $t = 0.6a$

$$\Rightarrow P\left(\sup \sum_{i=1}^{n} \alpha_i^2 - S_j^2 \geq t\right) \leq \frac{0.4a}{t} < 1$$

$$\Rightarrow P\left(\sum_{i=1}^{n} \alpha_i^2 - S_n^2 \geq 0.6 \sum_{i=1}^{n} \alpha_i^2\right) \leq \theta < 1$$

$$\Rightarrow P\left(S_n^2 \leq 0.4 \sum_{i=1}^{n} \alpha_i^2\right) \leq \theta < 1$$

$$\Rightarrow P\left(S_n^2 \geq 0.4a\right) \geq 1 - \theta \quad \forall n.$$

Recall

$$P\left(\sup_n S_n = +\infty\right) \overset{\text{symmetry by flipping } X_i}{=} P\left(\inf_n S_n = -\infty\right) \in \{0,1\}$$

$$= P\left(\sup_n |S_n| = +\infty\right) = P\left(\bigcap_{K>0} \bigcup_n \{|S_n| > K\}\right) = \lim_{K \uparrow \infty} P\left(\bigcup_n |S_n| > K\right)$$

~~$\geq P\left(\bigcap_{K>0}\right)$~~ for any K, choose n st $a = \sum_{i=1}^{n} \alpha_i^2 > \sqrt{K}^2$

$$\Rightarrow P'\left(|S_n| > K\right) \geq 1 - \theta$$ for all larger n

$$\Rightarrow P\left(\bigcup_n \{S_n > K\}\right) \geq 1 - \theta$$

Choose K close enough to the limit, and this shows

$$P\left(\sup_n |S_n| = +\infty\right) > 0$$

2nd solution:

Khinchin's inequality : Given $(\alpha_i)_i, (X_i)_i$ as above,

$$\mathbb{E}\left|\sum \alpha_i X_i\right| \geq \sqrt{\frac{1}{3}\sum \alpha_i^2}$$

Paley-Zygmund: If $Z > 0$

$$P(Z > \theta EZ) \geq (1-\theta)^2 \frac{E[Z]^2}{\cancel{Var} E(Z^2)}$$

Then

$$P(|S_n| > \theta E|S_n|) \geq (1-\theta)^2 \frac{E[|S_n|]^2}{E[S_n^2]} \geq (1-\theta)^2 C \frac{\cancel{E[S_n^2]}}{\cancel{E[S_n^2]}}$$

$$P\left(|S_n| > \frac{\theta}{\sqrt{3}}\sqrt{\sum \alpha_i^2}\right) \geq \frac{(1-\theta)^2}{3}$$

Pf of Khinchine: $X \geq 0$, $E[X^2] = E[X^\alpha X^\beta] \leq E[X^{\alpha p}]^{1/p} E[X^{\beta q}]^{1/q}$

let $\alpha p = 1$ $\quad \alpha + \beta = 2 \quad (2-\beta)p = 1 \quad 2p - \beta p = 1$

$\beta q = 4 \quad \beta \frac{p}{p-1} = 4 \quad 2p - 4(p-1) = 1$

$\Rightarrow -2p + 3 = 0 \quad p = 3/2 \quad q = \frac{3/2}{1/2} = 3$

$$E[X^2] \leq E[X]^{2/3} E[X^4]^{1/3} \Rightarrow E[X] \geq \frac{E[X^2]^{3/2}}{E[X^4]^{1/2}}$$

$E[X^2] = 1$

$\Rightarrow E\left[\sum \alpha_i \alpha_j \alpha_k \alpha_\ell X_i X_j X_k X_\ell\right]$ * $i=j$ $k=\ell$ or $i=j=k=\ell$ or $i=k$ $j=\ell$ or $i=\ell$

$$= 3\sum_{\substack{i=j\\k=\ell\\i\neq k}} \alpha_i^2 \alpha_k^2 + \sum_{i=1}^n \alpha_i^4 \leq 3\left(\sum_{i=1}^n \alpha_i^2\right)^2 = 3E[X^2]^2$$

$$\Rightarrow E|X| \geq \frac{E[X^2]^{3/2}}{E[X^2]} = E[X^2]^{1/2}$$

V. Nice.

Pf of Paley-Zygmund: Exercise for reader.